Time-domain inspiral templates for spinning compact binaries in quasi-circular orbits

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Abstract

We present a prescription to compute the time-domain gravitational wave (GW) polarization states associated with spinning compact binaries inspiraling along quasi-circular orbits. We invoke the orbital angular momentum $\boldsymbol{L}$ rather than its Newtonian counterpart $\boldsymbol{L}_{\rm N}$ to describe the orbits and the two spin vectors are freely specified in the source frame associated with the initial direction of the total angular momentum. We discuss the various implications of our approach.

1 Introduction

Gravitational waves (GWs) from coalescing compact binaries containing at least one spinning component are expected to be routinely detected by the second-generation laser interferometric detectors like advanced LIGO (aLIGO), Virgo and KAGRA [1]. The detection of GWs from such binaries and subsequent source characterization crucially depend on accurately modeling temporally evolving GW polarization states, $h_+(t)$ and $h_\times(t)$, from such binaries during their inspiral phase [2]. At present, $h_+(t)$ and $h_\times(t)$ associated with non-spinning compact binaries inspiraling along quasi-circular orbits have GW phase evolution accurate to 3.5PN order and amplitude

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corrections that are 3PN accurate [3]. Recall that the 3.5PN and 3PN orders correspond to corrections that are accurate to relative orders $(v/c)^7$ and $(v/c)^6$ beyond the 'Newtonian' estimates, where v and c are the orbital and light speeds, respectively. In the case of inspiraling compact binaries containing Kerr black Holes (BHs), it is desirable to employ temporally evolving $h_+(t)$ and $h_\times(t)$ that incorporate the spin effects very accurately and the dominant spin effect arises due to the general relativistic spin-orbit coupling that appear at the relative 1.5PN order for maximally spinning Kerr BHs [4, 5]. Following Ref. [5], we define the spin of a compact object as $\boldsymbol{S} = G\, m_{\rm co}^2\, \chi\, \boldsymbol{s}/c$, where $m_{\rm co}, \chi$ and $\boldsymbol{s}$ are its mass, Kerr parameter and a unit vector along $\boldsymbol{S}$, respectively and for a maximally spinning Kerr BH $\chi = 1$. We note that it is customary to employ the Newtonian orbital angular momentum $\boldsymbol{L}_{\rm N} = \mu\, \boldsymbol{r} \times \boldsymbol{v}$, where $\mu, \boldsymbol{r}$ and $\boldsymbol{v}$ are the reduced mass, orbital separation and velocity, respectively, to specify these quasi-circular orbits [5, 6].

In what follows we provide a prescription to generate the time-domain amplitude corrected $h_+(t)$ and $h_\times(t)$ for spinning compact binaries inspiraling along quasi-circular orbits, described by their orbital angular momenta. Note that the amplitude corrected $h_+(t)$ and $h_\times(t)$ refer to GW polarization states that are PN-accurate both in its amplitude and phase. We describe our approach to compute the fully 1.5PN accurate amplitude corrected expression for $h_\times(t)$ while invoking $\boldsymbol{L}$, the PN-accurate orbital angular momentum, to characterize the binary orbits. We symbolically obtain the additional 1.5PN order amplitude corrections to $h_\times(t)$ in comparison with equations (A3) in Ref. [6] that employ $\boldsymbol{L}_{\rm N}$ to describe the binary orbits. We also discuss certain implications of our approach while considering spin effects due to the leading order general relativistic spin-orbit coupling. Our attempt to perform GW phasing with the help of $\boldsymbol{L}$ is motivated by a number of observations (we term accurate modeling of temporally evolving GW polarization states as 'GW phasing'). First being Ref. [7] that provided a prescription to implement GW phasing for spinning compact binaries in inspiralling eccentric orbits in an accurate and efficient way. We are further influenced by the fact that it is customary to use precessional equation appropriate for $\boldsymbol{L}$ to evolve $\boldsymbol{L}_{\rm N}$ while incorporating the effects due to the dominant order spin-orbit coupling [5, 6, 8]. Finally, we note that a seminal paper that explored the inspiral dynamics of spinning compact binaries and the influences of precessional dynamics on $h_{+,\times}(t)$ employed $\boldsymbol{L}$ to describe their binary orbits [9]. Additionally, we specify the two spins in an inertial frame associated with the initial direction of the total angular momentum $\boldsymbol{j}_0$ while it is customary to invoke a non-inertial $\boldsymbol{L}_{\rm N}$-based orbital triad to

specify the two spins in the literature.

2 GW phasing for spinning binaries characterized by $\boldsymbol{L}$

We invoke, as noted earlier, the PN-accurate orbital angular momentum $\boldsymbol{L}$ to describe binary orbits and we specify at the initial epoch both the orbital and spin angular momentum vectors in an inertial frame associated with the initial direction of total angular momentum (see figure 1). We begin by presenting an expression for the cross polarization state having Newtonian (quadrupolar) order amplitude and the relevant expression reads [10]

$$\begin{aligned} h_\times|_{\rm Q}(t) = 2\,\frac{G\,\mu}{c^2\,R'}\,\frac{v^2}{c^2}\biggl\{&(1-\cos\iota)\,S_\theta\,\sin\iota\,\sin(\alpha-2\Phi) \\ &-(1+\cos\iota)\,S_\theta\,\sin\iota\,\sin(\alpha+2\Phi) \\ &-\frac{1}{2}(1+2\cos\iota+\cos^2\iota)\,C_\theta\sin(2\alpha+2\Phi) \\ &-\frac{1}{2}(1-2\cos\iota+\cos^2\iota)\,C_\theta\sin(2\alpha-2\Phi)\biggr\}\,, \end{aligned} \tag{1}$$

where R', S_θ and C_θ stand for the radial distance to the binary, $\sin\theta$ and $\cos\theta$, respectively. The dynamical angular variable Φ measures the orbital phase from the direction of ascending node in a plane perpendicular to $\boldsymbol{k}$ while its derivative is required to define $v^2/c^2 = (G\,m\,\dot\Phi/c^3)^{2/3}$. To obtain temporally evolving $h_\times|_{\rm Q}(t)$ associated with spinning compact binaries inspiraling along quasi-circular orbits, we pursue the following steps. First, we specify how the Eulerian angles α and ι that specify the orientation of $\boldsymbol{L}$ vary under the conservative orbital dynamics. This requires us to employ three differential equations that describe the precessional dynamics of the orbital and spin angular momenta and these differential equations contain an orbital like frequency ω [5]. The conservative evolution for Φ and $\dot\Phi$ are governed by the following differential equation

$$\dot\Phi = \omega - \cos\iota\,\dot\alpha\,, \tag{2}$$

where the orbital like frequency ω is defined by the relation $\omega = v/r$ that connects ω to the orbital separation and velocity. This PN-accurate equation arises from the vectorial expression for the orbital velocity in

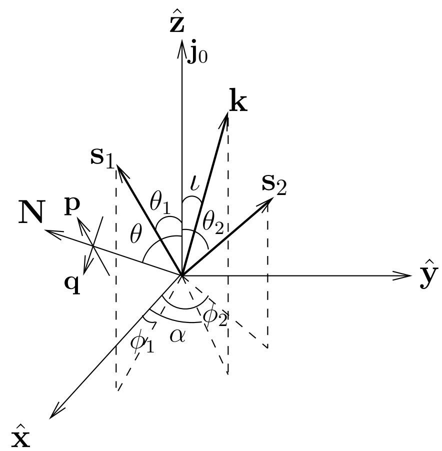

Figure 1: The inertial frame and the Cartesian coordinate system where the $\hat{\boldsymbol{z}}$ axis points along $\boldsymbol{j}_0$, the direction of total angular momentum at the initial epoch. We display the angles that characterize the orbital and spin angular momentum vectors, denoted by $\boldsymbol{k}, \boldsymbol{s}_1$ and $\boldsymbol{s}_2$, while the line of sight vector $\boldsymbol{N}$ is in the $x-z$ plane. The dashed lines depict the projections of various vectors on the $x-y$ plane.

a co-moving triad [5]. It is convenient to introduce a dimensionless parameter $x \equiv (G\,m\,\omega_{\rm orb}/c^3)^{2/3}$ such that the above equation reads $\dot{\Phi} = (x^{3/2}/(G\,m/c^3)) - \cos\iota\,\dot{\alpha}$. Thereafter, we impose the effects of gravitational radiation reaction on these differential equations by specifying how x, appearing in these differential equations, vary during the binary inspiral.

It turns out that we require to solve simultaneously the precessional equations for $\boldsymbol{k}, \boldsymbol{s}_1$ and $\boldsymbol{s}_2$, the unit vectors along $\boldsymbol{L}, \boldsymbol{S}_1$ and $\boldsymbol{S}_2$, to specify how α and ι vary under the conservative dynamics as the precessional dynamics of $\boldsymbol{L}, \boldsymbol{S}_1$ and $\boldsymbol{S}_2$ are intertwined. In figure 1, we display various angles that specify the orientations of these three unit vectors in the above inertial frame. The Cartesian components of these vectors are

$$\boldsymbol{k} = (\sin\iota\,\cos\alpha\,,\,\sin\iota\,\sin\alpha\,,\,\cos\iota)\,, \tag{3a}$$

$$\boldsymbol{s}_1 = (\sin\theta_1\,\cos\phi_1, \sin\theta_1\,\sin\phi_1, \cos\theta_1)\ , \tag{3b}$$

$$\boldsymbol{s}_2 = (\sin\theta_2\,\cos\phi_2, \sin\theta_2\,\sin\phi_2, \cos\theta_2)\ . \tag{3c}$$

The precessional equations for the above three unit vectors, extractable from

Ref. [4, 5], read

$$\dot{\boldsymbol{k}} = \frac{c^3}{Gm} x^3 \left\{ \delta_1 \, q \, \chi_1 \, (\boldsymbol{s}_1 \times \boldsymbol{k}) + \frac{\delta_2}{q} \chi_2 \, (\boldsymbol{s}_2 \times \boldsymbol{k}) \right\}, \tag{4a}$$

$$\dot{\boldsymbol{s}}_1 = \frac{c^3}{Gm} x^{5/2} \, \delta_1 \, (\boldsymbol{k} \times \boldsymbol{s}_1) \; , \tag{4b}$$

$$\dot{\boldsymbol{s}}_2 = \frac{c^3}{Gm} x^{5/2} \, \delta_2 \, (\boldsymbol{k} \times \boldsymbol{s}_2) \; , \tag{4c}$$

where $q = m_1/m_2$ and the symmetric mass ratio $\eta = \mu/m$ is required to define the quantities δ_1 and δ_2 ($\delta_{1,2} = \eta/2 + 3(1 \mp \sqrt{1-4\eta})/4$). We are now in a position to incorporate the effect of gravitational radiation damping and this is achieved by specifying the secular variations in x in the above differential equations. The PN-accurate differential equation for x arises from the usual energy balance arguments and may be obtained from equation (3.21) in Ref. [6].

We obtain numerically the time-domain GW polarization states for our binaries by simultaneously solving the differential equations for the Cartesian components of $\boldsymbol{k}, \boldsymbol{s}_1$ and $\boldsymbol{s}_2$ along with the PN-accurate equations for $\dot{\Phi}$ and dx/dt. This implies that we numerically solve a set of 11 differential equations and the resulting variations in α, ι, Φ and $\dot{\Phi}$ are implemented in equation (1) to obtain the temporally varying $h_\times|_{\rm Q}(t)$. A close inspection reveals that we require to specify four angles that provide the orientations of the two spins from $\boldsymbol{j}_0$ at the initial epoch to obtain $h_\times|_{\rm Q}(t)$. It should be noted that the two angles that specify the initial orientation of $\boldsymbol{k}$ from $\boldsymbol{j}_0$ are not independent variables. This is because one may point the total angular momentum along the z-axis at the initial epoch without the loss of any generality. This allowed us to equate the x and y components of $\boldsymbol{J} = \boldsymbol{L} + \boldsymbol{S}_1 + \boldsymbol{S}_2$ to zero at that epoch and therefore to obtain the initial estimates for ι and α. In our numerical integrations, the bounding values for x are given by $x_0 = 2.9 \times 10^{-4} \, (m' \, \omega_0)^{2/3}$ and $x_{\rm f} = 1/6$, where m' is the total mass of the binary in solar units and let $\omega_0 = 10 \, \pi$ Hz as customary for aLIGO along with $\Phi(x_0) = 0$. Finally, we note that the values of α and ι at every step of our numerical runs are obtained from the Cartesian components of $\boldsymbol{k}$ by the relations: $\alpha = \cos^{-1}(k_{\rm x}/\sqrt{k_{\rm x}^2 + k_{\rm y}^2})$ and $\iota = \cos^{-1}(k_{\rm z})$. Therefore, we require eight independent parameters to specify the inspiral dynamics of compact binaries with spinning components and these eight parameters are the four basic (constant) parameters namely $(m_1, m_2, \chi_1, \chi_2)$ along with the above four angular dynamical variables that specify the two

spin vectors in the inertial frame, namely (θ_1, ϕ_1) and (θ_2, ϕ_2) as displayed in figure 1.

In what follows we compare our approach with what is detailed in Ref. [6] that provided a way to obtain the time-domain ready-to-use GW polarization states for inspiraling compact binaries while incorporating all 1.5PN order spin effects both in amplitude and GW phase evolutions. Their approach differs from ours in two aspects. First, Ref. [6] employed $\boldsymbol{l} = \boldsymbol{L}_{\rm N}/|\boldsymbol{L}_{\rm N}|$ to characterize the orbital plane and following the earlier papers employed a precessional equation for $\boldsymbol{l}$ that is identical to our equation (4a) for $\dot{\boldsymbol{k}}$ while replacing $\boldsymbol{k}$ by $\boldsymbol{l}$. It is not very difficult to show that this is equivalent of using an orbital averaged expression for $\dot{\boldsymbol{l}}$ [5]. However, invoking an orbital averaged precessional equation for $\boldsymbol{l}$ leads to an undesirable feature that the coefficient of $\boldsymbol{l}$ in the expression for $\dot{\boldsymbol{n}}$ in the $(\boldsymbol{n}, \boldsymbol{\lambda} = \boldsymbol{l} \times \boldsymbol{n}, \boldsymbol{l})$ frame will not, in general, vanish. It is fairly straightforward to compute the time derivative of $\boldsymbol{n}$ and express it in the $(\boldsymbol{n}, \boldsymbol{\lambda}, \boldsymbol{l})$ frame as

$$\frac{d\boldsymbol{n}}{dt} = \left(\frac{d\Phi'}{dt} + \cos\iota' \, \frac{d\alpha'}{dt} \right) \boldsymbol{\lambda} + \left(\frac{d\iota'}{dt} \, \sin\Phi' - \sin\iota' \, \cos\Phi' \, \frac{d\alpha'}{dt} \right) \boldsymbol{l} \,, \quad (5)$$

where ι', α' and Φ' are the usual 3 Eulerian angles required to define $\boldsymbol{n}$, $\boldsymbol{\lambda}$ and $\boldsymbol{l}$ in the $\boldsymbol{j}_0$-based inertial frame. The fact that $\boldsymbol{l} \equiv \boldsymbol{n} \times \dot{\boldsymbol{n}}/|\boldsymbol{n} \times \dot{\boldsymbol{n}}|$ clearly demands that the coefficient of $\boldsymbol{l}$ in the above equation should be zero as also noted in Ref. [6]. However, if one employs equation (4a) to describe the precessional dynamics of $\boldsymbol{l}$ then it is possible to show with some straightforward algebra that

$$\sin\Phi' \, \frac{d\iota'}{dt} - \cos\Phi' \, \sin\iota' \, \frac{d\alpha'}{dt} = -\frac{c^3}{G\,m} \, x^3 \left\{ \delta_1 \, q \, \chi_1 \, (\boldsymbol{s}_1 \cdot \boldsymbol{\lambda}) + \frac{\delta_2}{q} \, \chi_2 \, (\boldsymbol{s}_2 \cdot \boldsymbol{\lambda}) \right\} , \quad (6)$$

as noted in Ref. [7]. It is not very difficult to conclude that the right hand side of above expression, in general, is not zero.

It turns out that $\dot{\boldsymbol{n}}$ having components along $\boldsymbol{l}$ leads to certain anomalous terms that contribute to the Φ' evolution at the 3PN order and this is within the consideration of higher order spin effects available in the literature. Note that the higher order spin effects are known to 3.5PN order while dealing with the spin-orbit interactions [11]. We demonstrate our observation by noting that the definitions $\boldsymbol{v} = r\,\dot{\boldsymbol{n}}$ and $v = r\,\omega$ imply that $\omega^2 = \dot{\boldsymbol{n}} \cdot \dot{\boldsymbol{n}}$. With the help of our Eq. (5) the expression for ω reads

$$\omega = \left(\frac{d\Phi'}{dt} + \cos\iota' \, \frac{d\alpha'}{dt} \right) + \frac{1}{2\,\dot{\Phi}'} \left(\frac{d\iota'}{dt} \, \sin\Phi' - \sin\iota' \, \cos\Phi' \, \frac{d\alpha'}{dt} \right)^2 , \qquad (7)$$

where $\dot{\Phi}'$ stands for $d\Phi'/dt$ and the higher order terms that are cubic in the time derivatives of ι' and α' are neglected. Invoking the fact that $\dot{\Phi}'$ at the Newtonian order is given by $x^{3/2}/(G\,m/c^3)$ and noting that the expression for $d\iota'/dt$ and $d\alpha'/dt$ arise from the 1.5PN order differential equation for $\dot{\boldsymbol{k}}$, we get

$$\dot{\Phi}' = \frac{c^3}{G\,m}\, x^{3/2} \left(1 + x^{3/2}\, A' + x^3\, B' \right) , \qquad (8)$$

such that A' and B' are given by

$$\begin{aligned} A' &= -\tfrac{\cos\iota'}{\sin\iota'} \Big\{ [\delta_1\, q\, \chi_1\, (\boldsymbol{s}_1 \cdot \boldsymbol{\lambda}) + \tfrac{\delta_2}{q}\, \chi_2\, (\boldsymbol{s}_2 \cdot \boldsymbol{\lambda})] \, \cos\Phi' \\ &\quad + [\delta_1\, q\, \chi_1\, (\boldsymbol{s}_1 \cdot \boldsymbol{n}) + \tfrac{\delta_2}{q}\, \chi_2\, (\boldsymbol{s}_2 \cdot \boldsymbol{n})] \, \sin\Phi' \Big\} , \qquad (9a) \\ B' &= -\tfrac{1}{2} \Big\{ \delta_1\, q\, \chi_1\, (\boldsymbol{s}_1 \cdot \boldsymbol{\lambda}) + \tfrac{\delta_2}{q}\, \chi_2\, (\boldsymbol{s}_2 \cdot \boldsymbol{\lambda}) \Big\}^2 . \qquad (9b) \end{aligned}$$

It should be evident that the B' terms appear at the 3PN order while A' terms enter $\dot{\Phi}$ expression at the 1.5PN order. It is not very difficult to infer that these B' terms arise due to the non-vanishing $\boldsymbol{l}$ component in the expression for $\dot{\boldsymbol{n}}$ and therefore are unphysical in nature. Therefore, these anomalous terms contribute to the Φ' evolution at the third post-Newtonian order and this is within the consideration of higher order spin effects currently available in the literature. It should be noted that these unphysical terms play no role in investigations in Ref. [6] that probed the leading order spin effects appearing at the 1.5PN order in the phase evolution.

Another consequence of invoking $\boldsymbol{k}$ to specify the binary orbit is the appearance of certain new 1.5PN order contributions to the amplitudes of h_+ and $h_\times$ in addition to what is provided by equations (A1), (A2) and (A3) in Ref. [6]. We recall that equations (A1), (A2) and (A3) in Ref. [6] provide the fully 1.5PN accurate expressions for h_+ and $h_\times$ while invoking $\boldsymbol{L}_{\rm N}$ to describe the binary orbits. The additional amplitude corrections to the GW polarization states arise mainly due to the fact that the component of $\boldsymbol{v}$ along $\boldsymbol{k}$ is of 1.5PN order. To demonstrate this point, we express $\boldsymbol{r}$ and $\boldsymbol{v} = d\boldsymbol{r}/dt$ in the inertial frame $(\hat{\boldsymbol{x}}, \hat{\boldsymbol{y}}, \hat{\boldsymbol{z}})$ associated with $\boldsymbol{j}_0$ with the help of the three usual Eulerian angles Φ, α and ι as displayed in figure 1. The relevant

expression for $\boldsymbol{r}$ may be written as $\boldsymbol{r} = r\boldsymbol{n}$ where $\boldsymbol{n} = (-\sin\alpha\,\cos\Phi - \cos\iota\,\cos\alpha\,\sin\Phi)\hat{\boldsymbol{x}} + (\cos\alpha\,\cos\Phi - \cos\iota\,\sin\alpha\,\sin\Phi)\hat{\boldsymbol{y}} + \sin\iota\,\sin\Phi\hat{\boldsymbol{z}}$. To show that $\boldsymbol{v}$ can have non-vanishing 1.5PN order terms along $\boldsymbol{k}$, we compute $d\boldsymbol{r}/dt$ in the co-moving frame defined by the triad $(\boldsymbol{n}, \boldsymbol{\xi} = \boldsymbol{k}\times\boldsymbol{n}, \boldsymbol{k})$ and this is easily achieved with the help of three rotations involving the three Eulerian angles appearing in the expression for $\boldsymbol{r}$ [12]. The resulting expression for $\boldsymbol{v}$ reads

$$\boldsymbol{v} = r\left(\frac{d\Phi}{dt} + \frac{d\alpha}{dt}\,\cos\iota\right)\boldsymbol{\xi} + r\left(\frac{d\iota}{dt}\,\sin\Phi - \sin\iota\,\cos\Phi\,\frac{d\alpha}{dt}\right)\boldsymbol{k}\,. \tag{10}$$

It is not very difficult to verify that $\boldsymbol{v}\cdot\boldsymbol{k} \neq 0$ while invoking equation (4a) for $\dot{\boldsymbol{k}}$ to evaluate the $\boldsymbol{k}$ component of $\boldsymbol{v}$. Moreover, the coefficient of $\boldsymbol{k}$ in the above expression for $\boldsymbol{v}$ is at the 1.5PN order. The $\boldsymbol{k}$ component of $\boldsymbol{v}$ enters the expressions for h_+ and $h_\times$ through the dot products $(\boldsymbol{p}\cdot\boldsymbol{v})\,(\boldsymbol{q}\cdot\boldsymbol{v})$ and $(\boldsymbol{p}\cdot\boldsymbol{v})^2 - (\boldsymbol{q}\cdot\boldsymbol{v})^2$ that are required to compute the PN-accurate expressions for h_+ and $h_\times$ (the vectors $\boldsymbol{p}$ and $\boldsymbol{q}$ are two unit vectors that lay in a plane perpendicular to $\boldsymbol{N}$). The resulting 1.5PN order amplitude corrections to $h_\times$ symbolically read

$$h_\times\Big|_{1.5\mathrm{PN}} = \frac{G\,\mu}{c^4\,R'}\,\frac{G\,m}{c^3\,\sqrt{x}}\left\{\frac{d\iota}{dt}\,\kappa_1(\iota,\alpha,\Phi,\theta) + \frac{d\alpha}{dt}\,\kappa_2(\iota,\alpha,\Phi,\theta)\right\}, \tag{11}$$

where the explicit expressions for κ_1 and κ_2 are available in Ref. [10]. Therefore, the fully 1.5PN order amplitude corrected expression for $h_\times$ associated with the spinning compact binaries in quasi-circular orbits, described by $\boldsymbol{L}$, is provided by equations (A3) along with the above equation (11). This statement also requires that the angular variables ι and α that appear in equations (A3) of Ref. [6] represent $\boldsymbol{k}$ rather than $\boldsymbol{l}$. Let us emphasize again that equations (A2) and (A3) in Ref. [6] indeed provide the fully 1.5PN accurate amplitude corrected h_+ and $h_\times$ for spinning compact binaries in circular orbits described by $\boldsymbol{L}_{\mathrm{N}}$.

Another aspect where we differ from Refs. [6, 8] is the way we specify the two spin and $\boldsymbol{k}$ vectors to perform GW phasing. In the literature, it is common to freely specify spin vectors in an orthonormal triad defined by using $\boldsymbol{l}$ [5, 8, 6]. In contrast, we freely specify the two spin vectors at the initial epoch in the inertial source frame associated with $\boldsymbol{j}_0$. This choice allowed us to specify the x and y components of $\boldsymbol{k}$ uniquely in terms of $\boldsymbol{s}_1, \boldsymbol{s}_2$ and other intrinsic binary parameters at the initial epoch by demanding that the x and y components of $\boldsymbol{J} = \boldsymbol{L} + \boldsymbol{S}_1 + \boldsymbol{S}_2$ should be zero at that epoch.

Therefore, we extract very easily the initial values of α and ι from the following expressions for the initial x and y components of $\boldsymbol{k}$

$$
\begin{aligned}
k_{\mathrm{x,i}} = \sin\iota\,\cos\alpha &= -\frac{G\,m^2}{c\,L_{\mathrm{i}}}\left\{X_1^2\,\chi_1\,\sin\theta_1'\,\cos\phi_1' + X_2^2\,\chi_2\,\sin\theta_2'\,\cos\phi_2'\right\}\,,\\
k_{\mathrm{y,i}} = \sin\iota\,\sin\alpha &= -\frac{G\,m^2}{c\,L_{\mathrm{i}}}\left\{X_1^2\,\chi_1\,\sin\theta_1'\,\sin\phi_1' + X_2^2\,\chi_2\,\sin\theta_2'\,\sin\phi_2'\right\}\,,
\end{aligned}
$$

where $\theta_1', \phi_1', \theta_2', \phi_2'$ are the values of $\theta_1, \phi_1, \theta_2, \phi_2$ at the initial epoch and L_{i} denotes the PN-accurate expression for $|\boldsymbol{L}|$ at initial orbital frequency.

For isolated unequal mass spinning compact binaries with $q > 3$ it should be advantageous to specify their spins in the inertial frame associated with $\boldsymbol{j}_0$. This is because $\theta_1(x_0)$ that provides the dominant spin orientation from $\boldsymbol{j}_0$ at x_0 is expected to lie in a smaller range for such binaries spiraling into x_0 due to the emission of GWs. A recent study reveals that the dominant BH spin orientation from $\boldsymbol{j}_0$ at x_0 is more likely to be $\leq 90°$ for angular momentum dominated unequal mass binaries ($|\boldsymbol{L}|(x_0) > S_1$) while $\theta_1(x_0) \leq 45°$ for unequal mass binaries having $S_1 > |\boldsymbol{L}|$ at x_0 [13]. These inferences require that these binaries inherit spin-orbit misalignments $\leq 160°$ from various astrophysical processes responsible for the formation of such unequal mass spinning binaries. The physical explanation for these conclusions is the observed alignment of $\boldsymbol{S}_1$ towards $\boldsymbol{j}_0$ due to the action of gravitational radiation reaction, detailed in Ref. [9]. However, it will be difficult to provide similar bounds for the dominant spin orientation if the spins are freely specified in a non-inertial orbital triad associated with $\boldsymbol{L}$ at x_0. In this case $\boldsymbol{k}\cdot\boldsymbol{s}_1$ provides the dominant spin orientation at x_0 and it is not difficult to show that $\boldsymbol{k}\cdot\boldsymbol{s}_1$ remains fairly constant as these unequal mass binaries with $q > 3$ spiral into x_0 from initial orbital separations $\leq 1000\,G\,m/c^2$ due to the emission of GWs.

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